A Characterization of the Maximin Rule in the Context of Voting

Ronan Congar & Vincent Merlin

CARE, Université de Rouen
&
CREM, CNRS and Université de Caen

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The Maximin Rule

In a voting context, when the preferences of the voters are described by linear orderings over a finite set of at least three alternatives, the Maximin rule orders the alternatives according to their minimal ranks in the voters' preferences. Consider the profile π with 5 voters and 5 alternatives :

$$\pi = \begin{pmatrix} a & a & c & d & e \\ b & b & a & b & c \\ c & c & b & a & a \\ e & e & e & e & b \\ d & d & d & c & d \end{pmatrix}$$

The minimal rank for a is 3 ; it is 2 for b and e, while it is 1 for d and c.

Basic model

- Let X denote a finite set of m alternatives, $m \geq 3$.
- Let $\mathcal{P}$ denote the set of all linear preference orderings over X and let $\mathcal{R}$ denote the set of all weak preference orderings over X. The restriction of the preference P to any set $Y \subset X$ is denoted by $P|_Y$.
- Let $\mathbb{N}$ denote the set of natural integers. Let $\mathcal{N}$ denote the set all non-empty finite subsets of $\mathbb{N}$:

$$\mathcal{N} = \{N \subset \mathbb{N} : 1 \leq \#N < \infty\}.$$

  A set $N \in \mathcal{N}$ is a finite **set of agents** (or voters).
- For $N \in \mathcal{N}$, denote by $\mathcal{P}^N$ the set of all **preference profiles** $\pi = (P_i)_{i \in N}$ such that $P_i \in \mathcal{P}$ for all $i \in N$. P_i is interpreted as the preference ordering of agent i.

Basic Model

Let $\mathcal{U} = \cup_{N \in \mathcal{N}} \mathcal{P}^N$ be the set of possible profiles.

- A **social welfare function (SWF)**

$$\begin{array}{rrcl} R: & \mathcal{U} & \to & \mathcal{R} \\ & \pi = (P_1, \ldots P_n) & \to & R(\pi) \end{array}$$

associates to every profile $\pi = (P_i)_{i \in N}$, $N \in \mathcal{N}$ a social ranking $R(\pi) \in \mathcal{R}$ on X.

- We write $xR(\pi)y$ if x is weakly preferred to y under social ranking $R(\pi)$. The symmetric (resp. asymmetric) part of $R(\pi)$ is denoted by $I(\pi)$ (resp. $P(\pi)$).

The Maximin Rule

- For all $N \in \mathcal{N}$, $\pi \in \mathcal{P}^N$, $i \in N$, and $x \in X$, the **rank** of x in terms of preference P_i is defined to be

$$r(x, P_i) = \#\{y \in X : xP_iy\} + 1.$$

- We also use the following shorthand notation

$$\min(x, \pi) = \min_{i \in N}(r(x, P_i)).$$

- For a preference $P \in \mathcal{P}$, we define $b^t(P) = \{x \in X : \ r(x, P) = t\}$, the alternative which has rank t in the preference P.
- $B^t(P) = \{x \in X : \ r(x, P) \leq t\}$ is the set of the t-bottom alternatives in P.

The Maximin Rule

The **Maximin ranking rule** $\underline{R}$. For all $N \in \mathcal{N}$, $\pi \in \mathcal{P}^N$, and $x, y \in X$,

$$x\underline{R}(\pi)y \Leftrightarrow \min(x, \pi) \geq \min(y, \pi)$$

The Maximin rule orders the alternatives according to their minimal rank in the preference profile.

Neutrality

For all permutation μ of X onto itself, for all $P \in \mathcal{P}$, define $\mu(P)$ as $xPy \Leftrightarrow \mu(x)\mu(P)\mu(y)$. For $\pi \in \mathcal{U}$, also define $\mu(\pi)$ as $\mu(\pi) = (\mu(P_i))_{i \in N}$

Neutrality (N). For all $N \in \mathcal{N}$, $\pi \in \mathcal{P}^N$, for all permutation μ of X onto itself, and all $x, y \in X$,

$$xR(\pi)y \Leftrightarrow \mu(x)R(\mu(\pi))\mu(y)$$

The familiar neutrality condition requires a symmetric treatment of all the alternatives.

Neutrality

Let $\mu(a) = d$, $\mu(b) = b$, $\mu(c) = c$, $\mu(d) = a$ and $\mu(e) = e$.

$$\pi = \begin{pmatrix} a & a & c & d & e \\ b & b & a & b & c \\ c & c & b & a & a \\ e & e & e & e & b \\ d & d & d & c & d \end{pmatrix}, \mu(\pi) = \begin{pmatrix} \mathbf{d} & \mathbf{d} & c & \mathbf{a} & e \\ b & b & \mathbf{d} & b & c \\ c & c & b & \mathbf{d} & \mathbf{d} \\ e & e & e & e & b \\ \mathbf{a} & \mathbf{a} & \mathbf{a} & c & \mathbf{a} \end{pmatrix}$$

$$\begin{array}{rl} & a \, \underline{P}(\pi) \, b \, \underline{I}(\pi) \, e \underline{P}(\pi) \, d \, \underline{I}(\pi) \, c \\ \Rightarrow & d \, \underline{P}(\mu(\pi)) \, b \, \underline{I}(\mu(\pi)) \, e \, \underline{P}(\mu(\pi)) \, a \, \underline{I}(\mu(\pi)) \, c \end{array}$$

Unanimity

The following axiom is the well known unanimity requirement.

Unanimity (U). For all $N \in \mathcal{N}$, $\pi \in \mathcal{U}$ and $x, y \in X$ such that xP_iy for all $i \in N$: $xP(\pi)y$.

If all voters prefer one alternative to another the former is ranked strictly above than the latter in the social ranking.

$$\pi = \begin{pmatrix} a & a & c & d & e & e & a \\ b & b & a & a & a & c & d \\ c & c & b & b & b & a & e \\ e & e & e & e & d & b & b \\ d & d & d & c & c & d & c \end{pmatrix}$$

We get here $a\underline{P}(\pi)b$.

Let $L(x,P) = \{y \in X : xRy\}$ be the **lower contour set** of x at preference P. Symmetrically, $U(x,P) = \{y \in X : yRx\}$ is the **upper contour set** of x at preference P. The following axiom has been introduced by Barberà and Dutta (1982) to characterize a 'prudent' voting rules.

Top Invariance (TI). For all $N \in \mathcal{N}$, $\pi, \pi' \in \mathcal{P}^N$ and $x \in X$ such that :

(i) $\forall i \in N, U(x,P_i) = U(x,P'_i)$

(ii) $\forall i \in N,\ P_i|_{L(x,P_i)} = P'_i|_{L(x,P'_i)}$,

we obtain $xR(\pi)y \Leftrightarrow xR(\pi')y$

The social ranking between x and y is unaffected by modifications of the individual preferences above x.

$$\pi = \begin{pmatrix} a & b & c & d & e \\ b & a & a & b & c \\ c & c & b & a & a \\ e & e & e & e & b \\ d & d & d & c & d \end{pmatrix}, \pi' = \begin{pmatrix} \mathbf{b} & b & c & d & e \\ \mathbf{c} & a & \mathbf{b} & b & \mathbf{b} \\ \mathbf{a} & c & \mathbf{a} & a & \mathbf{c} \\ \mathbf{e} & e & e & e & \mathbf{a} \\ d & d & d & c & d \end{pmatrix}$$

In π', we have changed the preferences above alternative d. The new ranking is

$$bP(\pi')aP(\pi')eP(\pi')dI(\pi')c$$

instead of

$$aP(\pi)bI(\pi)eP(\pi)dI(\pi)c.$$

Top invariance is a weak form of Arrow's IIA and Muller and Satterthwaite's Strong Positive Association. It also can be compared to Maskin Monotonicity.

Duplication

The hallmark axiom of the decision under complete ignorance literature is a duplication property (see Arrow and Hurwicz (1972), Milnor (1954)). Duplication captures the very essence of the notion of complete ignorance. If the outcomes of an alternative in two states of the world are identical and if the outcomes of another alternative are also identical in the same two states, this axiom declares the distinction between these two states irrelevant for the ranking of the alternatives involved. We propose here a definition of Duplication in the voting context.

Duplication

Duplication (D). For all $N \in \mathcal{N}$, $\pi \in \mathcal{P}^N$, let $j \in \mathbb{N}$ such that $j \notin N$. Let π' be a profile on $N \cup \{j\}$ such that $P_i = P_j$ for some $i \in N$. Then, a SWF R satisfies Duplication iff $R(\pi) = R(\pi')$.

Whenever a new voter join the population, his preference has no impact on the social ranking if this preference was already present in the initial profile.

$$\pi = \begin{pmatrix} a & a & c & d & e \\ b & b & a & b & c \\ c & c & b & a & a \\ e & e & e & e & b \\ d & d & d & c & d \end{pmatrix}, \pi' = \begin{pmatrix} a & c & d & e \\ b & a & b & c \\ c & b & a & a \\ e & e & e & b \\ d & d & c & d \end{pmatrix}$$

$$\pi" = \begin{pmatrix} a & a & c & d & e & e & e & e \\ b & b & a & b & c & c & c & c \\ c & c & b & a & a & a & a & a \\ e & e & e & e & b & b & b & b \\ d & d & d & c & d & d & d & d \end{pmatrix}$$

Duplication

Is the Duplication axiom reasonable property ? There are context for which the Duplication condition makes sense.

As already pointed out by Brams and Kilgour (2001), the maximin is a way a identify possible compromises, rather than designating a clear winner. Having an outrageous majority for one candidates then makes no sense if the objective is to find a compromise or even to protect a minority opinion.

The duplication axiom may also be meaningful in multicriteria decision analysis, where each ordering represents either the opinion of an expert or the recommendation of some criteria. It emphasizes the fact that the divergences in opinion are more important that the number of experts behind each judgement.

Duplication

For all $N \in \mathcal{N}$, permutation σ of N onto itself and $P \in \mathcal{P}^N$, let $\sigma(\pi) = (P_{\sigma(i)})_{i \in N}$.

Anonymity (A). For all $N \in \mathcal{N}$, $\pi \in \mathcal{P}^N$ and permutation σ of N onto itself : $R(\sigma(\pi)) = R(\pi)$.

The familiar anonymity condition requires all the individuals' preferences to be treated symmetrically.

Theorem (1)

A SWF satisfies Anonymity whenever it satisfies Duplication.

A result for three alternatives

Theorem (2)

If $\#X = 3$, a SWF satisfies Neutrality, Duplication, Top Invariance and Unanimity, if and only if it is the Maximin rule.

Let $B(\pi) = \{x \in X : r(x, P_i) = m \; for \; some \; i \in N\}$. It is the set of alternatives which are ranked last by at least one voter.

Lemma (1)

Consider a SWF which satisfies Neutrality, Duplication and Top Invariance. $\forall x, y \in B(\pi)$, $xI(\pi)y$. Moreover, if the SWF also satisfy Unanimity, $\forall z \in X \setminus B(\pi)$ and $\forall x \in B(\pi)$, $zP(\pi)x$.

A result for three alternatives

- If $B(\pi) = \{a, b, c\}$, by Lemma (1), $xI(\pi)y$ for any $x, y \in \{a, b, c\}$. Thus, R coincides with $\underline{R}$.
- If $B(\pi) = \{a, b\}$, by Lemma (1), the rule coincide with the maximin $\underline{R}$.
- If $B(\pi) = \{a\}$, by Duplication, we can reduce the profile to π^1, π^2, or π^3 :

$$\pi^1 = \begin{pmatrix} b \\ c \\ a \end{pmatrix}, \pi^2 = \begin{pmatrix} c \\ b \\ a \end{pmatrix}, \pi^1 = \begin{pmatrix} b & c \\ c & b \\ a & a \end{pmatrix}$$

The rule coincide with $\underline{R}$ in any case.

Independence of the axioms

The **Maximax**. The maximax rule orders the alternatives according to their maximal rank in the preference profile. Formally, for all $N \in \mathcal{N}$, $\pi \in \mathcal{P}^N$, $x, y \in X$, $xR(\pi)y$ if and only if

$$\max(x, \pi) \geq \max(y, \pi)$$

The maximax rule satisfies Duplication, Neutrality and Unanimity but fails Top Invariance.

The **Anti Maximin (or Minimin)**. Formally, for all $N \in \mathcal{N}$, $\pi \in \mathcal{P}^N$, $x, y \in X$, $xR(\pi)y$ if and only if

$$\min(x, \pi) \leq \min(y, \pi)$$

The Anti Maximin rule satisfies Duplication, Neutrality, and Top Invariance but fails Unanimity.

Independence of the axioms

The **Antipurality rule** gives one point to each alternative each time it is not ranked last ; it it ranks the alternative according to their antiplurality score. Antiplurality satisfies TI and N, but fails U and D.

The **Antiplurality Run-Off** rule selects first the top two candidates on the basis of the antiplurality rule. The top two candidates are ranked according to the majority rule. Antiplurality Run-off satisfies TI, U and N, but fails D.

The **Maximin with an alphabetical tie breaking rule** atisfies all the axioms but Neutrality.

Another rule for $\#X = 4$

The reduced profile $\hat{\pi}$ is deduced from π. It contains all the preferences which are present in π and only them, but only once. Consider the domains of profiles $\mathcal{D} \subset \mathcal{U}$ where the same

two alternatives are ranked last or next to the last.

The *Majority-Maximin* works as follows :

- If $\pi \in \mathcal{P}^N \setminus \mathcal{D}$, then apply the Maximin criteria.
- If $\pi \in \mathcal{D}$, use the majority criterion on the reduced profile $\hat{\pi}$ for the top two alternative, and the maximin for the two bottom ranked candidates.

Another rule for $\#X = 4$

$$\pi = \begin{pmatrix} c & d & d \\ d & c & c \\ a & a & b \\ b & b & a \end{pmatrix}$$

The Maximin rule gives

$$d\underline{I}(\pi)c\underline{P}(\pi)a\underline{I}(\pi)b$$

while the Majority-Maximin rule proposes

$$dP^{\star}(\pi)cP^{\star}(\pi)I^{\star}(\pi)b.$$

The four axioms do not uniquely characterize the maximin with more than three alternatives.

A fifth axiom

A classical axiom in voting theory is the separability property that Smith proposed in 1973 for the characterization of the scoring rules (see also Young (1973,1975)).

For all $N, N' \in \mathcal{N}, N \cap N = \emptyset, \pi \in \mathcal{P}^N, \pi' \in \mathcal{P}^{N'}$, denote by $\pi + \pi'$ the **combined profile** $(P_i)_{i \in N \cup N'} \in \mathcal{P}^{N \cup N'}$.

Separability (S). For all $N, N' \in \mathcal{N}$, $N \cap N = \emptyset$, $\pi \in \mathcal{P}^N$, $\pi' \in \mathcal{P}^{N'}$ and all $x, y \in X$:

(i) $xP(\pi)y$ and $xR(\pi')y$ imply $xP(\pi + \pi')y$;

(ii) $xI(\pi)y$ and $xI(\pi')y$ imply $xI(\pi + \pi')y$.

A fifth Axiom

Unfortunately, the Maximin does not satisfy the Separability condition, as it fails to meet point (i).

$$\pi = \begin{pmatrix} a & b \\ b & a \\ c & c \end{pmatrix}, \pi' = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \pi + \pi' = \begin{pmatrix} a & b & a \\ b & a & b \\ c & c & c \end{pmatrix}$$

We get $a\underline{I}(\pi)b$ and $a\underline{P}(\pi')b$, but $a\underline{I}(\pi + \pi')b$.

A fifth Axiom

All we need is a weaker property : **Weak Separability (WS).**

For all $N, N' \in \mathcal{N}, N \cap N = \emptyset$, $\pi \in \mathcal{P}^N$, $\pi' \in \mathcal{P}^{N'}$ and all $x, y \in X$:

(i) $xP(\pi)y$ and $xP(\pi')y$ imply $xP(\pi + \pi')y$;

(ii) $xI(\pi)y$ and $xI(\pi')y$ imply $xI(\pi + \pi')y$.

Whenever two complementary profiles π and π' select *exactly the same ranking* for a pair of alternatives under a given SWF, the SWF applied directly to the combined profile also selects the same ranking.
It is easy to check that the Maximin rule satifies Weak Separability.

The Main Theorem

Theorem (3)

If $\#X \geq 4$, a SWF satisfies Neutrality, Duplication, Top Invariance, Unanimity and Weak Separability, if and only if it is the Maximin rule.

- TI The Maximax rule
- U The Negative Maximin (or Minimin) rule
- N The Maximin with an alphabetical tie breaking rule.
- D The rule which applies Antiplurality and breaks ties by unanimity.
- WS The rule which applies the maximin to all profile except those in which all voters share the top two alternatives. In the later case the rule applies Reduced Majority to rank the top two alternatives, the other alternatives are ranked according to the Maximin.

A Comparison with Antiplurality

The Antiplurality rule selects the alternatives with the minimal number of last ranks.

Bottom Unanimity (BU). Let $\pi \in \mathcal{P}^N$ and $x \in X$ such as yP_ix, $\forall i \in N$, $\forall y \in X,\ y \neq x$. Then, the social welfare function R satisfies Bottom Unanimity, if and only if $yP(\pi)x$ $\forall y \in X,\ y \neq x$.

Whenever an alternative is unanimously ranked last, it is also ranked last in collective ranking.

Theorem (Merlin 1996)

A SWF satisfies Neutrality, Anonymity, Separability, Top Invariance and Bottom Unanimity, if and only if it is the Antiplurality rule.

TAB.: A comparison between Maximin and Antiplurality

Antiplurality	Maximin Rule
Neutrality	Neutrality
Top Invariance	Top Invariance
Anonymity	Duplication
Bottom Unanimity	Unanimity
Separability	Weak Separability

Dual Results for Maximax

Bottom Invariance (B). For all $N \in \mathcal{N}$, $\pi, \pi' \in \mathcal{P}^N$ and $x \in X$ such that : (i) $\forall i \in N, L(x, P_i) = L(x, P'_i)$ and (ii) $\forall i \in N$, $P_i|_{U(x,P_i)} = P'_i|_{U(x,P'_i)}$,

$$xR(\pi)y \Leftrightarrow xR(\pi')y.$$

Bottom Invariance asserts that modifications of individual preferences below alternative x do not change its collective preferences compared to any other alternatives.

Theorem

If $\#X \geq 4$, then a SWF satisfies Neutrality, Duplication, Bottom Invariance, Unanimity and Weak Separability, if and only if it is the Maximax rule. When $\#X = 3$, it is characterized by Neutrality, Duplication, Bottom Invariance and Unanimity.

A characterization for the Plurality ruke

The Plurality rule ranks the alternatives according to their number of first places.

Top Unanimity (TU). Let $\pi \in \mathcal{P}^N$ and $x \in X$ such as xP_iy $\forall i \in N$, $\forall y \in X,\ y \neq x$. A social welfare function R satisfies *Top Unanimity*, if and only if $xP(\pi)y$, $\forall y \in X,\ y \neq x$.

Whenever an alternative is unanimously ranked first, it is also ranked last in collective ranking.

Theorem (Merlin 1996)

A Social Welfare Function satisfies Neutrality, Anonymity, Separability, Bottom Invariance and Top Unanimity, if and only if it is the Plurality rule.

TAB.: A comparison between Maximax and Plurality

Plurality	Maximax
Bottom Invariance	Bottom Invariance
Neutrality	Neutrality
Anonymitty	Duplication
Top Unanimity	Unanimity
Separability	Weak Separability

Conclusion and open questions

1. We have provided a rather simple characterization of the Maximin principle in the context of voting.
2. The characterization we have provided has been inspired by the existing literature on scoring rules.
3. We have adapted the duplication property in the context of voting
4. We have suggested contexts where the duplication property is desirable : multicriteria decision-making, social choice, compromise, protection of minorities
5. We have provided a comparison between the maximin and the antiplurality rule (‘prudent’ voting rules) and dual results

Conclusion and open questions

1. It should be possible to extend the result to social choice correspondences, by using Saari's Weak Consistency instead of Weak Separability.
2. In the context of SCC, Moulin have defined the concept of *prudent voting*, and Barberà and Dutta have proposed the concept of *protective behavior* to describe the strategies of risk averse voters facing uncertainty. Is the maximin implementable with prudent voting and protective behavior ?
3. It seems to difficult to extend the results to profiles of weak ordering. How to interpret a completly indifferent voter ?
4. What about Leximin ? Is it possible to extend the Maniquet and Sen's results ?